Journal of Statistical Physics, Vol. 30, No. 1, 1983

Preliminaries to the Ergodic Theory of Infinite-Dimensional Systems: A Model of Radiant Cavity

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Received June 2, 1982

We discuss a number of mathematical results that are relevant to the statistical mechanics of a model of radiant cavity in which the electromagnetic field interacts with a nonlinear charged oscillator. In particular, we show that energy equipartition in the sense of Jeans would exclude local exponential instability of orbits; it would also prevent the existence of significant finite invariant measures on a given energy surface. A phase space of infinite total energy is defined, and an invariant measure in it is built, for which different modes of the field are statistically independent.

KEY WORDS: Stochasticity; infinite systems; black body.

1. INTRODUCTION

The ergodic properties of conservative classical Hamiltonian systems with a finite number N of degrees of freedom are reasonably well understood. Even if precise ergodic properties can be established only in a very limited number of cases, nevertheless ergodic theory provides a general useful frame for understanding the statistical behavior of classical physically interesting systems with a finite number of particles.

Matters are different when dealing with an infinite number of degrees of freedom, a case which, on the other hand, possesses a definite physical interest. Indeed, while in finite-dimensional systems some of the mathematical hardware is somewhat natural—all norms are equivalent, intuitive

Work supported by CNR Grant No. 80.02384.

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0022-4715/83/0100-0195$03.00/0

invariant measures exist, etc.—so that we are sometimes justified in brushing it under the rug, this is not so for infinite-dimensional systems. For instance, even finding suitable phase spaces and establishing existence and uniqueness of solutions by no means constitute trivial steps.

Equilibrium of matter with radiation constitutes the most prominent problem in this category. Within the framework of classical physics a famous solution was proposed by Jeans.[1] Jeans's conjecture is based on the generalization of the equipartition theorem of classical statistical mechanics to this infinite-dimensional case. The legitimacy of such an extension is, however, far from unquestionable. Should this "theorem" be false then the classical problem of ether and matter in equilibrium would have to be completely reexamined.

Several years ago, the interesting idea was proposed that the classical approach to the problem of black-body radiation might be fruitfully reconsidered in the light of recent achievements of ergodic theory and nonlinear dynamics,[2–5] in order to check the *a priori* assumptions on which the Rayleigh–Jeans law rests, i.e., to check

i. whether a trend towards some sort of statistical equilibrium is exhibited by a radiant cavity in which a nonlinear mechanism of energy exchange between the normal modes is at work.

ii. whether, in the eventual equilibrium state, equipartition of energy holds.

The problem is a difficult one to deal with in general terms; a specific model was devised by Bocchieri, Crotti, and Loinger[2] which proved amenable to numerical analysis.[2–6]

All these numerical investigations, adopted to a varying extent the strategy of inquiring the "stochasticity" of the BCL model by looking for one or another of the peculiarities by which this quality manifests itself in dynamical sytems with finitely many degrees of freedom. However, in the absence of an ergodic theory of infinite-dimensional Hamiltonian systems comparable to the one available for the finite-dimensional case, this approach may turn out to be exceedingly naive: the more so the subtler the techniques being borrowed from the finite-dimensional theory.

As a matter of fact, as was pointed out on more than one occasion, "stochasticity" is a quite vague notion, the widespread, informal use of which is usually justified by the underlying rigorous categories provided by ergodic theory[7] but looks pointless in the absence even of the necessary preliminaries of the latter. As a definite example the Liapounov characteristic exponents (LCE) are a prominent tool in establishing the stochastic properties of finite-dimensional systems. It is far from obvious, however, that they remain equally significant in the infinite-dimensional case.

In the present paper, by discussing a specific model, we point out some characteristic difficulties of infinite-dimensional systems. More precisely we present here some results that provide the mathematical preliminaries to the ergodic theory of the BCL model, namely, (i) identification of various phase spaces, i.e., of functional spaces in which the equations of motion define a well-set Cauchy problem, (ii) construction of invariant measures in these phase spaces.

Actually we are able to set up a phase space appropriate to describe the time evolution of finite energy states. We cannot answer the question whether an invariant measure on the energy surface exists. Rather unexpectedly existence of such an invariant measure would imply Jeans's conjecture to be false. Thus ergodicity in this sense and equipartition are in contrast. One consequence of this fact is that it removes one of the assumptions which appear to be essential for the LCE machinery to be meaningful in statistical mechanics. In fact we show that energy equipartition would imply the vanishing of LCE. In other words, positive exponents preclude energy equipartition, in sharp constrast to the finite-dimensional case. In Section 5 we discuss a phase space in which the generic state has an infinite total energy and we describe an invariant measure in it which is a generalization of the canonical measure for a system of finitely many oscillators; a noteworthy aspect is that, in this case, the motion of the mechanical oscillator is, with probability 1, nondifferentiable. The metrical transitivity of this measure is left as an open problem which, in our opinion, might prove a rewarding task for future investigation. In the concluding Section 6, we comment briefly on other aspects related to our previous work and on some questions that can be raised about the significance of the BCL model.

2. JEANS'S CONJECTURE AND BCL MODEL

. . . Thus, excluding the impossible case of a system having infinite energy, we see that the temperature of a perfectly structureless medium ought to be invariably zero. Whenever an exchange of energy takes place between the medium and a material system placed in it, the medium must always gain energy and the rest of the system must always lose energy. The final state can therefore only be one in which all of the energy of the material system has been transmitted to the medium, and both are at zero temperature.[1]

Stated in "dynamical" terms, Jeans's expectation is that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E_n(t)\,dt=0$$

$E_n(t)$ being the total energy of any mechanical or field oscillator. The time average is evaluated along an orbit on a given energy surface, and the result should hold for "most" reference orbits.

The equipartition theorem of classical statistical mechanics rests upon the existence of a "natural" ergodic finite invariant measure. If one *requires* that time averages be independent of the reference orbit up to sets of (Lebesgue) measure zero, it follows that these averages are given by microcanonical ensemble averages: in particular, equipartition holds.

The legitimacy of Jeans's extrapolation to the infinite-dimensional case is by no means obvious. It would be interesting to know whether, also in this case, there is one more or less "natural" assumption of what the null sets are, and whether this assumption, together with the requirement that time averages be independent of the chosen orbit up to null sets is sufficient to single out one ergodic finite invariant measure on the given energy surface.

However, this would *exclude* Jeans's expectation. In fact, this expectation is inconsistent with the very existence of a finite invariant measure having the prescribed null sets. For, suppose that

$$\lim_{n\to\infty} \frac{1}{T}\int_0^T E_n(t)\,dt = 0$$

for μ almost all initial data, with μ some finite (normalized) invariant measure. Then

$$\lim_{T\to\infty} \frac{1}{T}\int d\mu \int_0^T E_n(t)\,dt = \overline{E}_n$$

the ensemble average of E_n: hence, $\overline{E}_n = 0\ \forall n$. However, $\sum_n E_n = E$ the total energy. If μ is supported by the manifold of total energy E, by monotone convergence we get $0 = \sum_n \overline{E}_n = \overline{\sum_n E_n} = E$, which is absurd.

The above remark suggests that such concepts as ergodicity, equipartition, etc. will not play here the same role as in the finite-dimensional case (and we will see that the same holds for LCE as well). To clarify the situation in full generality is of course a difficult problem and we will study a specific model formerly proposed by Bocchieri, Crotti, and Loinger.[(2)] It will be enough for our purposes to recall that the model describes the radiation trapped in between two perfectly reflecting plane mirrors and interacting with a nonlinear charged oscillator in the form of a charged plane placed midway and sliding parallel to the mirrors. In a suitable reference frame the relevant equations are

$$\begin{aligned} \frac{\partial^2 A}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} &= -\frac{4\pi}{c}\sigma\delta(x)\dot{z} \\ m\ddot{z} &= -\frac{\sigma}{c}\frac{\partial A(0,t)}{\partial t} - \alpha z^3 - m\omega_0^2 z \end{aligned} \tag{1}$$

with the boundary conditions $A(\pm l, t) = 0$, $2l$ being the distance between the mirrors. Here z is the displacement of the oscillator, A is the z component of the vector potential, and σ and α measure the charge and the nonlinearity of the oscillator.

In Ref. 4 we have shown formally that (1) is an infinite-dimensional Hamiltonian system, with the energy given by[4]

$$E = \frac{1}{2} m\dot{z}^2 + \frac{1}{4}\alpha z^4 + \frac{1}{2}\int_{-l}^{l}\left[\frac{1}{4\pi}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4\pi c^2}\left(\frac{\partial A}{\partial t}\right)^2\right]dx + \frac{1}{2} m\omega_0^2 z^2 \tag{2}$$

3. THE FINITE-ENERGY PHASE SPACE

A preliminary step to the study of the statistical behavior of the model, is setting the question of existence and uniqueness of the initial value problem for Eqs. (1). This is not a trivial problem and apart from the mere existence statement its solution will provide qualitative information about the character of the orbits, which will be of use in the sequel. One rather obvious choice is to consider finite-energy solutions first: this leads to the familiar energy-space of the system. In fact we will show in Theorems 1–3 that for any assignment of initial conditions $(A(x,0), A_t'(x,0), z(0), \dot{z}(0))$ such that E defined in Eq. (2) is finite, the initial-value problem is well posed and the orbits lie on an infinite-dimensional manifold Σ_E in a suitable Hilbert space.

To formulate the Cauchy problem in a Hilbert space setting, it will be convenient to rewrite the differential equations (1) as a first-order system:

$$\begin{aligned} \dot{z} &= y \\ \frac{\partial A}{\partial t}(t,x) &= v(t,x) \\ \dot{y} &= -\omega_0^2 z - \frac{\alpha}{m} z^3 - \frac{\sigma}{cm} v(0,t) \\ \frac{\partial v}{\partial t}(t,x) &= c^2 \frac{\partial^2 A(t,x)}{\partial x^2} + 4\pi c\sigma y \delta(x) \end{aligned} \tag{3}$$

We take up the linear case $\alpha = 0$ first. Let $\mathbb{C}$ and $L^2(I)$ have the usual meaning and denote by $H_0^1(I)$ the space of absolutely continuous functions on $I = [-l, l]$ having a square summable derivative and vanishing at $\pm l$. Let X be the complex linear space:

$$X = \mathbb{C} \times H_0^1(I) \times \mathbb{C} \times L^2(I) \tag{4}$$

[4] Eq. 2 represents the energy, not the Hamiltonian. In the latter an interaction term would appear.

A point in X is then a quadruplet $\xi = (z, u, y, v)$; X is a Hilbert space when given the scalar product

$$(\xi', \xi'')_X = \omega_0^2 \bar{z}' z'' + \frac{1}{4\pi} \int_{-l}^{l} \frac{d\bar{u}'}{dx} \frac{du''}{dx}\, dx + \bar{y}' y'' + \frac{1}{4\pi c^2} \int_{-l}^{l} \bar{v}' v\, dx \qquad (5)$$

The Cauchy problem for Eqs. (3) with $\alpha = 0$, can be formulated as the problem of solving in X the equation

$$\begin{aligned} \frac{d\xi}{dt} &= T\xi \\ \xi(0) &= \xi_0 \end{aligned} \qquad (6)$$

where the linear operator T is formally defined by

$$T\xi = \left(y, v, -\omega_0^2 z - \frac{\sigma}{cm} v(0), c^2 \frac{\partial^2 A}{\partial x^2} + 4\pi c\sigma y \delta(x) \right) \qquad (7)$$

It will be presently seen that the domain $D(T)$ of T can be so chosen that T is a skew-adjoint linear operator in X.

Lemma. Let $a(\xi, \xi^*)$ be defined on $X \times X$ by

$$a(\xi, \xi^*) = (z\bar{y}^* - \bar{z}^* y) + \frac{1}{4\pi c^2} \int_{-l}^{l} (u\bar{v}^* - \bar{u}^* v) + \frac{\sigma}{cm} \left[z\bar{u}^*(0) - u(0)\bar{z}^* \right] \qquad (8)$$

then $a(\cdot, \cdot)$ is a bounded skew-symmetric form on $X \times X$. If G is defined in X by

$$a(\xi, \xi^*) = (G\xi, \xi^*)_X \qquad \forall \xi, \quad \xi^* \in X \qquad (9)$$

then G is a (bounded, skew-adjoint) compact and invertible linear operator (see Appendix A).

By computation one sees that $G\xi = \hat{\xi}$ is the unique element in X satisfying

$$\begin{aligned} \hat{z} &= -\frac{1}{\omega_0^2} \left[y + \frac{\sigma}{cm} u(0) \right] \\ -\frac{\partial^2 \hat{u}}{\partial x^2} &= -\frac{1}{c^2} v + \frac{4\pi\sigma}{c} z\, \delta(x) \\ \hat{y} &= z \\ \hat{v} &= u \end{aligned} \qquad (10)$$

Define $D(T) = \mathrm{Rg}(G)$. Then the above shows that

$$c^2 \frac{\partial^2 \hat{u}}{\partial x^2} + 4\pi\sigma c\hat{y}\, \delta(x) = v \qquad (11)$$

and $\hat{v}(0) = u(0)$. Since $v \in L^2(I)$ and $u(0)$ makes sense by the continuity of $u \in H_0^1(I)$, this shows that the right-hand side of Eq. (7) is well defined whenever $\xi \in D(T)$, and that T is inverse to G.

Theorem 1 follows from the above.

Theorem 1. T is skew-adjoint and possesses a pure point spectrum. Therefore T generates a unitary group e^{Tt} which solves the linear problem. To each eigenvector u_n of T there corresponds a constant of the motion:

$$K_n = |c_n| = |(\xi, u_n)_X|$$

Also $t \to e^{Tt}$ is an almost periodic function; this means that every solution of the equations of motion is almost periodic.

The set of eigenvectors $\{u_n\}$ and the corresponding set of eigenvalues Ω_n are reported in Appendix A. Consider now the nonlinear case $\alpha \neq 0$. System (3) will now be formally rewritten as

$$\frac{d\xi}{dt} = T\xi + W(\xi) \tag{12}$$

where W is the nonlinear operator in X defined by

$$W(\xi) = -\frac{\alpha}{m}|(K_0, \xi)|^2(K_0, \xi)K_1 \tag{13}$$

where $K_0 = (1, 0, 0, 0)$ and $K_1 = (0, 0, 1, 0)$. The following is an existence and uniqueness theorem for the solution of Eq. (12) which will provide a basis for all subsequent considerations.

Theorem 2. For any $\xi_0 \in X$ Eq. (12) has a unique weak solution $z(t) \in C_0(\mathbb{R}, X)$ such that $\xi(0) = \xi_0$. By weak solution of (12) here it is meant that $\forall \eta \in D(T), (\xi(t), \eta)_X$ is C^1 in t, and

$$\frac{d}{dt}(\xi(t), \eta)_X = -(\xi(t), T\eta)_X + (W(\xi(t)), \eta)_X \tag{14}$$

The proofs of this theorem and of Theorem 3 below are given in Appendix B. One minor difficulty is that $K_1 \notin D(T)$ so that W does not map $D(T)$ into itself. This is why Theorem 2 is not completely covered by standard results.

One important consequence of this theorem is that, defining $c_n(\xi) = (u_n, \xi)$ for $\xi \in \mathcal{H}$, $c_n(\xi(t))$ is C^1 in t and that

$$\begin{aligned} c_n'(t) &= i\Omega_n c_n + \alpha\nu_n|\varphi(\xi(t))|^2\varphi(\xi(t)), \qquad \nu_n = (K_1, u_n) \\ \lambda_n &= (u_n, K_0), \qquad \varphi(\xi) = \Sigma\lambda_n c_n(\xi) \end{aligned} \tag{15}$$

The infinite system of Eqs. (15) is the formulation of the dynamics of the BCL model to be used in Section 5. This formulation and Theorem 2 define the dynamics of the BCL model in the space $\mathcal{H}$ of complex sequences $\{c_n\}$ s.t. $\Sigma|c_n|^2 < +\infty$.

The result concerning continuous dependence of the solution on the initial data is as follows:

Theorem 3. Let $U_t\xi_0 = \xi(t)$ be the solution with initial condition ξ_0 at time t. $\{U_t\}$ is a group of nonlinear operators in $\mathcal{H}$. Then $\forall t$, U_t is uniformly continuous on bounded subsets of $\mathcal{H}$.

4. EQUIPARTITION EXCLUDES EXPONENTIAL INSTABILITY

In this section we shall make a partial use of the above results to investigate one important aspect of the statistical behavior of the model.

As is well known, in the case of finite-dimensional Hamiltonian systems, positivity of the maximum Lyapounov exponent for almost all initial conditions implies strong statistical properties.[8] We shall investigate how far this carries through the BCL infinite-dimensional model.

Restricting to real solutions,[5] the variational equation associated with (12) reads[9]

$$\frac{d\xi}{dt} = T\xi + B(t)\xi \tag{16}$$

where the bounded linear operator $B(t)$ is defined by

$$B(t) = -3\alpha z^2(t)(\cdot, K_0)K_1$$

and $z(t)$ is read along a chosen reference orbit. Attempting a generalization of the LCEs to this infinite-dimensional case involves the investigation of[6]

$$\lim_{t\to\infty} \frac{1}{t} \log\|\xi(t)\| \tag{17}$$

$\xi(t)$ being the solution of Eq. (16) corresponding to a generic initial vector $\xi(0) \in \mathcal{H}_r$, where $\mathcal{H}_r$ is the real invariant subspace of $\mathcal{H}$.

In the following we will show that if equipartition holds then the limit (17) is always zero. In fact we have the following proposition:

[5] The adoption of a complex phase in Section 3 was mainly motivated by the reason that it made it easy to derive Eqs. (15), which describe the cavity as a numerable set of interacting oscillators.

[6] In Ref. 6 no definition of what should be meant by LCE in the infinite-dimensional case is given. However, the procedure adopted therein shows that the authors consider LCEs to be given by $\lim_{N\to\infty} l_N$, l_N being the LCEs of a finite (N-degrees of freedom) approximation of the cavity. This definition differs from (14) by the interchange of the limits $N\to\infty$, $t\to\infty$. This interchange is hardly acceptable, if only because $\lim_{N\to\infty}\lim_{t\to\infty}$ does *not* depend on the norm chosen in each $2N$-dimensional phase space, while $\lim_{t\to\infty}\lim_{N\to\infty}$ obviously does.

Proposition. Let

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T E_0(t)\,dt = 0$$

along a given reference orbit, E_0 being the energy of the mechanical oscillator. Then

$$\lim_{t\to\infty} \frac{1}{t}\log\|\xi(t)\| = 0, \qquad \forall \xi(0) \neq 0$$

along the same orbit.

For the proof we prepare the following comments:

Let $\xi(t) = e^{Tt}\varphi(t)$. Then, if $\varphi(t)$ is a strong solution of the equation

$$\frac{d\varphi}{dt} = e^{-Tt}B(t)e^{Tt}\varphi = V(t)\varphi \tag{18}$$

with $\varphi(0) = \xi(0)$, one easily sees that $\xi(t)$ is a weak solution of Eq. (16) with $\xi(0)$ as initial condition. Existence (and uniqueness) for solutions of (18) is a standard result; in fact, $V(t)$ is a continuous map from R into $B(\mathcal{H}_r)$ (the bounded linear operators in $\mathcal{H}_r$ with the operator norm), since, we know from Theorem 2 that $z(t) \in C^1$. Moreover one has the estimate[10]

$$\|\varphi(t) - \varphi(0)\| \leqslant e^{t\|V\|_t}t\|V\|_t\|\varphi(0)| \tag{19}$$

where

$$\|V\|_t = \sup_{0\leqslant s\leqslant t} \|V(s)\|_{B(\mathcal{H}_r)}$$

Proof of the Proposition. Choose a $\tau > 0$ and let $I_n = [n\tau, (n+1)\tau]$. Let $z_n(t), \varphi_n(t), V_n(t)$ be defined in $[0,\tau]$ by $z_n(t) = z(t+n\tau)$, $\varphi_n(t) = \varphi(t+n\tau)$, $V_n(t) = V(t+n\tau)$. Denoting by $\overline{E}_0^K$ the energy of the mechanical oscillator $\frac{1}{2}mz^2 + \frac{1}{2}m\omega_0^2z^2 + \frac{1}{4}\alpha z^4$ averaged over I_K, we shall see below that

$$\|V_K\|_\tau \leqslant \lambda(\tau)\overline{E}_0^K \tag{20}$$

where

$$\lambda(\tau) = 12\frac{\alpha}{m}\left(\frac{1}{\omega_0^2} + \tau^2\right)$$

Then, by Eq. (19) one has

$$\big|\|\varphi_K(\tau)\| - \|\varphi_K(0)\|\big| \leqslant e^{\tau\lambda(\tau)E}\tau\lambda(\tau)\overline{E}_0^K\|\varphi_K(0)\|$$

where E is the total energy. Since $\varphi_K(0) = \varphi_{K-1}(\tau)$, we have

$$\left|1 - \frac{\|\varphi_K(\tau)\|}{\|\varphi_{K-1}(\tau)\|}\right| \leqslant e^{\tau\lambda(\tau)E}\tau\lambda(\tau)\overline{E}_0^K$$

Let $\alpha_K = \|\varphi_K(\tau)\|/\|\varphi_{K-1}(\tau)\|$; by choosing τ conveniently {e.g., such that $\exp[\tau\lambda(\tau)E]\tau\lambda(\tau)E \leqslant 1/2$} we have $|1-\alpha_K| \leqslant 1/2$ so that

$$|\ln\alpha_K| \leqslant 2|1-\alpha_K| \leqslant 2\exp\left[\tau\lambda(\tau)E\right]\tau\lambda(\tau)\overline{E}_0^K$$

and finally

$$\begin{aligned}\left|\frac{1}{n\tau}\ln\frac{\|\varphi(n\tau)\|}{\|\varphi(0)\|}\right| &= \left|\frac{1}{n\tau}\ln\frac{\|\varphi_{n-1}(\tau)\|}{\|\varphi(0)\|}\right| \\ &= \left|\frac{1}{n\tau}\ln\frac{\|\varphi_{n-1}(\tau)\|}{\|\varphi_{n-2}(\tau)\|}\cdot\frac{\|\varphi_{n-2}(\tau)\|}{\|\varphi_{n-3}(\tau)\|}\cdots\frac{\|\varphi_1(\tau)\|}{\|\varphi_0(\tau)\|}\right| \\ &= \left|\frac{1}{n\tau}\sum_{K=1}^{n-1}\ln\alpha_K\right| \leqslant \frac{2}{n}\lambda(\tau)\exp\left[\tau\lambda(\tau)E\right]\sum_{K=1}^{n-1}\overline{E}_0^K \\ &= 2\lambda(\tau)\exp\left[\tau\lambda(\tau)E\right]\frac{1}{n\tau}\int_0^{n\tau}E_0(s)\,ds \to 0 \qquad \text{as} \quad n\to\infty\end{aligned}$$

Since $\|\xi(t)\| = \|\varphi(t)\|$ because e^{Tt} is unitary, the statement follows.

As to the proof of (20) notice that

$$\|V_K\|_\tau = \|B_K\|_\tau \to 3\alpha \sup_{s\in[0,\tau]} |z_K^2(s)|$$

By Poincaré's inequality for C^1 functions

$$\|V_K\|_\tau \leqslant 3\alpha\left\{\frac{2}{\tau}\int_0^\tau z^2\,dt + 2\tau\int_0^\tau \dot{z}^2\,dt\right\}$$

and fitting in various constants gives (20) and concludes the proof. ∎

Thus a positive LCE and equipartition cannot coexist. It is of some interest to notice that numerical experiments[6] failed to reveal this fact. The point is, of course, that any numerical computation implies some sort of truncation to a finite number N of degrees of freedom; the task of estimating large time limits becomes then particularly tricky. The above result brings to the forefront in an impressive manner the kind of problems mentioned in the Introduction. Many other evolution equations of mathematical physics define dynamical systems with infinite-dimensional phase space which would be interesting to study in the framework of ergodic theory. In some cases it may well happen that the underlying phenomenology justifies some kind of cutoff: for example, solutions of the Navier–Stokes equations are attracted by finite-dimensional subspaces so that the physicist feels justified in performing some finite-mode truncation.

Instead, the electromagnetic field possesses an actual infinity of degrees of freedom; otherwise the ultraviolet catastrophe would be no more than a harmless mathematical pathology. Therefore, it is very likely that in any computation the correct procedure $\lim_{t\to\infty}\lim_{N\to\infty}$ would lead to different results from the "thermodynamical limit" $\lim_{N\to\infty}\lim_{t\to\infty}$. On the other hand, numerical computations can give direct information only on some kind of $\lim_{N\to\infty}\lim_{t\to\infty}$; with the result that any phenomenon reminiscent of the "stochastic transition" discovered in this way may well be of no relevance to the actual infinite-dimensional system.

To summarize, both Liapounov exponents and stochastic transition are concepts whose precise relevance to the problem at hand is far from clear and should be handled with care.

5. REMARKS ON THE CASE OF INFINITE ENERGY

That the system of matter and ether inside an enclosure might possess an infinite energy was regarded by Jeans as "an impossible case."[(1)] Since then, the development of quantum field theory has been accustoming physicists to the idea that the electromagnetic field can be a very pathological mathematical object.

In this section we will show that in the present case, two circumstances at least suggest that a phase space of infinite energy might be the proper setting for a discussion of ergodic properties of the BCL model. These are that one such space is actually available, i.e., the equations of motion of the cavity define in this space a group of (nonlinear) continuous evolution operators—and that in this space, unlike the space of finite energy, one invariant measure (at least) can be found by the procedure of finite-dimensional approximation.

In other words, the statistical mechanics in this space is the limit $N\to\infty$ of the statistical mechanics of a system of N oscillators.

In order to define this phase space, consider in $\mathcal{H}$ the norm $\|\xi\|_{-1}^2 = \sum\Omega_n^{-2}|c_n(\xi)|^2$, and let $\mathcal{H}_{-1}$ be the space obtained by completing $\mathcal{H}$ in this norm. Points in $\mathcal{H}_{-1}$ may be considered to be the representatives of states of the cavity, for which $\sum\Omega_n^{-2}|c_n|^2 < \infty$.

The evolution of states in the space $\mathcal{H}_{-1}$ is described by the following theorem:

Theorem 4. The group $\{U_t\}$ in $\mathcal{H}$ extends in a unique way to a group $\tilde{U}_t$ of nonlinear operators in $\mathcal{H}_{-1}$. $\tilde{U}_t$ is $\mathcal{H}_{-1}$ continuous in t; moreover $c_n(t) = c_n(\tilde{U}_t x)$ obeys Eqs. (15) for all n and $\forall x \in \mathcal{H}_{-1}$.

Proof. The estimate (B.3) holds in the $\mathcal{H}_{-1}$ norm as well: in fact, all the estimates leading to (B.3) can be rewritten in the $\mathcal{H}_{-1}$ norm. To see this, one needs only take notice that φ is a bounded functional on $\mathcal{H}_{-1}$:

$$|\varphi(\xi)| = \Big|\sum_j \lambda_j c_j\Big| \leqslant \Big(\sum_j |\lambda_j|^2 \Omega_j^2\Big)^{1/2} \Big(\sum_j |c_j|^2 \Omega_j^{-2}\Big)^{1/2} = \|TK_0\|\, \|\xi\|_{-1} \quad (21)$$

From (B.3) one sees that U_t is $\mathcal{H}_{-1}$-uniformly continuous on $\mathcal{H}_{-1}$-bounded subsets of $\mathcal{H}$, and the first statement of the theorem follows. The continuity of $\tilde{U}_t$ with respect to t is also proved, by the estimate (B.3) in the $\mathcal{H}_{-1}$-norm. In fact this estimate shows that if $\xi_n \in \mathcal{H}$ converges to $x \in \mathcal{H}_{-1}$ in the $\mathcal{H}_{-1}$-norm, then $U_t \xi_n$ converges to $\tilde{U}_t x$ uniformly with respect to t in compact sets of the line.

Now, observe that φ and c_n extend, as bounded functionals, to $\mathcal{H}_{-1}$. Then, since $c_n(\xi_K(t)) \to_{K\to\infty} c_n(x(t))$ uniformly on bounded intervals, the last part of the theorem follows. $\mathcal{H}_{-1}$ has been obtained, regarding the cavity as an assembly of infinitely many oscillators and allowing their total energy to be infinite. As a matter of fact, there is another way in which one may represent the BCL model as a system of infinitely many oscillators; this is also the way referred to in Ref. 4. It consists in expanding $A(x,t)$ in a sine series and considering the complex amplitudes of the expansions as oscillator coordinates. In this way, one finds the following system of infinitely many equations:[4]

$$\begin{aligned}
\dot{x}_n &= \omega_{2n+1} y_n \\
\dot{z} &= \omega_0 y + 2\omega_0 \epsilon \sum_0^\infty \omega_{2n+1}^{-1} x_n \\
\dot{y}_K &= -\omega_{2n+1} x_n - \epsilon \omega_0^{-1} \dot{z} \\
\dot{y} &= -\omega_0 x
\end{aligned} \qquad (22)$$

where $\omega_{2n+1} = (2n+1)\pi c/2l$ and ϵ is a parameter containing σ, m, c, l. The energy of the cavity is now given by

$$E = mc^2 \Big\{ \sum_n \big[|x_n|^2 + |y_n|^2\big] + |x|^2 + |y|^2 \Big\}$$

For all states $\xi \in X$ the amplitudes x_n, y_n are well-defined quantities, that describe the state of the oscillators of the "free" cavity ($\gamma = 0$, $\alpha = 0$).

The sum inside the curly brackets in this expression of E defines in X (or in $\mathcal{H}$) a norm equivalent to the one that we have actually used.

Referring to the oscillators described by (22) might look more convenient, because the material oscillator is just one of them.

Nevertheless, the space obtained by completing X in the norm

$$\sum_n n^{-2} \big\{ |x_n|^2 + |y_n|^2 \big\} + |x|^2 + |y|^2$$

is not the same as $\mathcal{H}_{-1}$; in fact, y does *not* extend as a bounded functional to $\mathcal{H}_{-1}$ since, as already remarked, K_1—i.e., the vector defining the velocity of the mechanical oscillator—$\notin D(T)$. In this new space no theorem analogous to Theorem 4 can be given. The physical reason seems to be that, as the energy of the field grows to infinity, the mechanical oscillator is forced into a motion of a very irregular nature, for which no instantaneous velocity y can be defined. Thus its kinematics (if not its statistics) looks like that of a Brownian particle.

In $\mathcal{H}_{-1}$ a $\tilde{U}_t$-invariant measure can be obtained, by the procedure of finite-dimensional approximation. To get such an approximation, consider system (15) truncated to N oscillators:

$$\begin{aligned} c_n'(t) &= 0, && |n| > N \\ c_n'(t) &= i\Omega_n c_n + i\alpha\omega^{-1}\Omega_n\lambda_n \left|\sum_{-N}^{N} \lambda_K c_K\right| \sum_{-N}^{2N} \lambda_K c_K, && |n| \leqslant N \end{aligned} \tag{23}$$

Equations (A.4) have been used to get $\nu_n = i\omega^{-1}\lambda_n$. Using the same Eqs. (A.4) one sees that the λ_n's are real numbers, so that, by writing $c_n = p_n + i\Omega_n q_n$ with real p_n, q_n one gets the system

$$\begin{aligned} p_n' &= -\Omega_n^2 q_n - \alpha\omega^{-1}\Omega_n\lambda_n \left\{ \sum_{k,l=-N}^{N} \lambda_K\lambda_l[p_k p_l + \Omega_k\Omega_l q_k q_l] \right\} \sum_{K=-N}^{N} \lambda_K \Omega_K q_K \\ q_n' &= p_n + \alpha\omega^{-1}\lambda_n \left\{ \sum_{K,l=-N}^{N} \lambda_K\lambda_l[p_K p_l + \Omega_k\Omega_l q_K q_l] \right\} \sum_{K=-N}^{N} \lambda_K p_K \quad |n| \leqslant N \end{aligned} \tag{24}$$

$$p_n' = q_n' = 0, \qquad |n| > N$$

This is a Hamiltonian system, with the Hamiltonian function

$$H_N = \frac{1}{2}\sum_{-N}^{N}(p_n^2 + \Omega_n^2 q_n^2) + \frac{\alpha}{2\omega}\left(\sum_{K,l=-N}^{N} \lambda_K\lambda_l[p_K p_l + \Omega_K\Omega_l q_K q_l]\right)^2$$

Thus (24) defines a flow $U_N(t)$ in $\mathcal{H}_{-1}$ thanks to usual results of the theory of ordinary DEs. A measure on $\mathcal{H}_{-1}$, invariant for this flow is easily found: suppose now that the $p_n q_n$ are random variables defined on some probability space, and that they are distributed according to the density

$$\mathcal{C}_n \exp\{-\sigma^{-2} H_N(q_{-N}\cdots q_N; p_{-N}\cdots p_N)\} \prod_{|j|>N} W_0(p_j, q_j) \tag{25}$$

$\mathcal{C}_n$ being a normalization constant, and W_0 an arbitrary normalized density with a finite variance. Since the expectation value of $\sum\Omega_n^{-2}|c_n|^2$ is finite with probability 1 (25) defines a Borel measure $\mu_{N,\sigma}$ on $\mathcal{H}_{-1}$, that is obviously invariant for the flow $U_N(t)$.

The following results can now be proved:

Proposition 1. As $N \to \infty$, the measure $\mu_{N,\sigma}$ converges weakly to a measure μ_σ on $\mathcal{H}_{-1}$. The characteristic functional of μ_σ is

$$M_\sigma(\mathbf{z}) = c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dj\, d\theta\, M(\gamma,\theta)\exp\left[-\frac{\theta^2}{2}\|(\gamma+i\theta)K_0-\xi\|^2\right], \qquad \xi \in \mathcal{H} \tag{26}$$

$M(\gamma,\theta)$ being the Fourier transform of $e^{-(x^2+y^2)^2}$.

Proposition 2. $\forall x \in \mathcal{H}_{-1}$, $\lim_{N\to\infty} U_N(t)x = \tilde{U}_t x$ (in $\mathcal{H}_{-1}$-norm).

Proposition 3. μ_σ is $\tilde{U}_t$ invariant.

All these results are proved in Appendixes C and D. The measure μ_σ might also be obtained as the limit $N \to \infty$ of microcanonical N-oscillator measures with energy N_σ^2. Some simple properties of the measure μ_σ become evident in the representation provided by the variables x_n, y_n. Referring to these variables we see that, despite the fact that μ_σ describes a possible equilibrium ensemble for an infinite system of interacting oscillators, it looks like the one describing perfect oscillator gas.[7] In fact one has the following proposition:

Proposition 4. All the x_n, y_n's with the exception of y extend, as linear functionals, to $\mathcal{H}_{-1}$. Any two of them with $n > 0$ are independent Gaussian random variables on the probability space $(\mathcal{H}_{-l}, \mu_\sigma)$.

Proof. To prove the first statement, consider x_n, y_n and x as bounded linear functionals on $\mathcal{H}$, and let $K_n^{(x)}, K_n^{(y)}, K_0$ be unit vectors spanning their respective Cokers. By using (A.4) one easily sees that all these vectors lie in $D(T)$, so that $\forall n \in \mathcal{H}$

$$|x_n(u)| = \left|\sum_j \bar{c}_j(K_n^{(x)})c_j(u)\right| \leqslant \left[\sum_j \Omega_j^2|c_j(K_n^{(x)})|^2\right]^{1/2}\left[\sum_j \Omega_j^{-2}|c_j(u)|^2\right]^{1/2}$$
$$= \|TK_n^{(x)}\|\,\|u\|_{-1}$$

Let $\xi = aK_n + bK_m$, $n \neq m$, $n \neq 0$, $m \neq 0$ in (22):

$$M_\sigma(aK_n + bK_m) = c\int\int_{-\infty}^{\infty} d\gamma\, d\theta\, M(\gamma,\theta)\exp\left[-\frac{\sigma^2}{2}(\gamma^2+\theta^2)\right]$$
$$\times\ \exp\left[-\frac{\sigma^2}{2}(a^2+b^2)\right]$$
$$= \exp\left[-\frac{\sigma^2}{2}a^2\right]\exp\left[-\frac{\sigma^2}{2}b^2\right]$$

[7] Of course, a much more surprising result of this kind is met in the classical approach to paramagnetism.

Since M_σ, as a function of a, b is just the joint characteristic functional of the random variables (K_n, x) and (K_m, x) with x a sample in $(\mathcal{H}_{-1}, \mu_0)$ this completes the proof.

6. CONCLUSIONS

In this paper we have attempted to provide some mathematical clarifications on the dynamics of a model of a radiant cavity. These clarifications are essential if one aims to pass from the provisional stage at which the investigations on this model have been set up, to a more sophisticated and rigorous kind of analysis. The upshot of our analysis is that certain mathematical constructions taken for granted in the finite-dimensional case so as to be considered a part of the usual scenery, such as the existence of a natural invariant measure, must be critically reexamined when infinite degrees of freedom are involved. For example, Jeans's conjecture which appeared up to now as a logical extension to the infinite-dimensional case of what has been considered a persuasive stochasticity criterion, i.e., energy equipartition, cannot be matched with a meaningful ergodic theory. Therefore if, despite this, tendency to equipartition is kept as a signpost of stochastic behavior of the model, then recourse to other more sophisticated stochasticity criteria of the ergodic theory for finite systems such as the Lyapounov exponents must be automatically excluded. In particular we have shown here that equipartition excludes the possibility of finding some positive LCE. We would like to stress also that numerical evaluation of LCE in the present case, involves double limits which very likely cannot be interchanged: this problem appears to us as overwhelmingly difficult.

It is fit to mention that the model has been also tested in Ref. 6 by means of a different "stochasticity parameter." However, as far as we know, in all works in which this parameter p has been introduced[(11)] and used[8] no justification at all was given regarding its relation to stochastic behavior. This parameter measures somehow the energetic involvement of one or more "oscillators" in the overall motion of the system and is defined in such a way as to take values ranging from zero (no energy exchange) to one (complete energy exchange). While it is obviously true that ergodicity implies $p = 1$, it is evident that the possible utility of this parameter rests on the validity of the converse statement. Now it is conceivable that by adding an integrable perturbation to an integrable system the "oscillators" associated with the unperturbed normal modes will share their energy to a less or larger extent depending on the spread of the new normal modes over the unperturbed ones. Therefore it looks hardly acceptable that this pa-

[8] See papers (7–11) of Ref. 6.

rameter, *per se*, could distinguish between integrable and stochastic behavior.[9]

Given all this, our analysis does not answer the question of whether, as far as one considers states of finite total energy, a tendency to statistical equilibrium, in the sense of ergodic theory at least, can take place or not. We may only recall the results of our previous numerical computations[(4)] which suggested a negative answer. This would mean that the toroidal structure of phase space as described in Theorem 1 essentially persists under perturbation. In other terms we are led to think that some kind of Kolmogorov–Arnold–Moser theorem is valid for the problem at hand.

It is possible that finite N-oscillators approximations of the BCL model exhibit a "Stochastic transition" at same value E_N of their energy. In this event one should not disregard the possibility that $E_N \to \infty$ as $N \to \infty$.

If this were true[10] one should expect good ergodic properties of the flow described in Section 5 in the phase space $\mathcal{H}_{-1}$. The BCL model might then provide a new dynamical model of Brownian motion (of the mechanical oscillator).[11]

ACKNOWLEDGMENT

We are indebted to Y. Sinai for helpful discussions.

APPENDIX A

We prove the Lemma of Section 2.

It is convenient to factor X as

$$X = \hat{H}_0^1 \times \hat{L}^2 = \left[\mathbb{C} \times H_0^1(I)\right] \times \left[\mathbb{C} \times L^2(I)\right]$$

[9] As matter of fact the regularity exhibited by the numerical experiments illustrated in Fig. (1) of Ref. 6 may be interpreted in terms of resonance energy exchange between the field nth normal mode with frequency $\omega_n = (\pi c/2l)n$ and the nonlinear mechanical oscillator with frequency $\omega(E)$. Indeed, using the same notations as in Ref. 6 these two frequencies are equal when $\epsilon \div n/\gamma$; this might explain Figs. 1–3.

[10] Some of the results of Ref. 6 might be indicative of this fact even though they have been presented as evidence of a different type of behavior not investigated here. However, the only investigations made in this direction, which could dispose of an effective control of N, failed to reveal any stochastic behavior (Ref. 12). The computational setup of Refs. 4 and 6 are hybrid in the sense that they do not correspond to any definite N-oscillator truncation of the equations of motion.

[11] An exact dynamical model of Brownian motion is actually available for a particle linearly interacting with a one-dimensional, semi-infinite continuum: see, e.g., Ref. 13. In this model an essential role is played by the possibility of introducing a "translation representation" for the dynamics of the *linear, infinite* system. An analogous role might be played here by any eventual K-system property of the *nonlinear, finite* system.

and write $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)$ as $\xi=(\xi^1,\xi^2)$, $\xi^1\in\hat{H}_0^1$, $\xi^2\in\hat{L}^2$. Also notice that the function

$$\mu(x)=\begin{cases}\frac{1}{2}x+l/2, & -l\leqslant x\leqslant 0\\ -\frac{1}{2}x+l/2, & 0\leqslant x\leqslant l\end{cases}$$

belongs to $H_0^1(I)$ and is such that

$$(\mu,v)_{H_0^1}=v(0)\qquad\forall v\in H_0^1(I)$$

It is now easy to check that

$$a(\xi,\eta)=(\xi^1,\eta^2)_{\hat{L}^2}-(\xi^2,\eta^1)_{\hat{L}^2}+\frac{\sigma}{c}\xi_1(\mu,\eta_2)_{H_0^1}-\frac{\sigma}{c}\bar{\eta}_1(\xi_1,\mu)_{H_0^1}\quad\text{(A.1)}$$

Clearly $a(\xi,\eta)=-a(\eta,\xi)$. By an elementary case of Rellich's theorem the embedding $H_0^1(I)\to L^2(I)$ is compact (and so is therefore $\hat{H}_0^1\to\hat{L}^2$).

Boundedness of $a(\cdot,\cdot)$ follows from estimates of the type

$$|(\xi^1,\eta^2)_{\hat{L}^2}|\leqslant m\|\xi^1|_{\hat{L}^2}\|\eta^2\|_{\hat{H}}\leqslant m\|\xi\|_X\|\eta\|_X$$

(m the embedding constant) and

$$|\xi_1(\mu,\eta_2)|\leqslant|\xi_1|\,\|\mu\|_{H_0^1}\|\eta_2\|_{H_0^1}\leqslant c\|\xi\|_X\|\eta\|_X$$

Define G in X by

$$a(\xi,\eta)=(G\xi,\eta)_X\qquad\forall\xi,\eta\in X\qquad\text{(A.2)}$$

By the above, G is bounded and skew-adjoint: $G^+=-G$. To see compactness of G, let $\{\xi_n\}$ be a sequence in X converging weakly to $\hat{\xi}$ so that $G\xi_n\to G\hat{\xi}$ weakly. By (A.1)

$$\|G\xi_n\|^2=a(\xi_n,G\xi_n)\qquad\text{(A.3)}$$

The sequence $\{(G\xi_n)^2\}$ converges weakly in $\hat{H}_0^1$ and so strongly in $\hat{L}^2$ to $(G\hat{\xi})^2$. Therefore

$$\left(\xi_n^1,(G\xi_n)^2\right)_{\hat{L}^2}\to\left(\hat{\xi}^1,(G\hat{\xi})^2\right)_{\hat{L}^2}$$

and similarly

$$\left(\xi_n^2,(G\xi_n)^1\right)_{\hat{L}^2}\to\left(\hat{\xi}^2,(G\hat{\xi})^1\right)_{\hat{L}^2}$$

Now (A.3) and (A.1) show that $\|G\xi_n\|\to\|G\hat{\xi}\|$, thus $G\xi_n$ converges strongly to $G\hat{\xi}$ and so G is compact.

Since form $a(\cdot,\cdot)$ is nondegenerate, zero is not in the point spectrum of G and G is invertible. Therefore T possesses a complete orthonormal set of eigenvectors. Among these one has the trivial cases $z=0,y=0$ and u, v eigensolutions of the free vibrating string problem of odd parity. The nontrivial eigenvalues $i\Omega$ are found as the roots of

$$\tan\left(\frac{\omega l}{c}\right) = \frac{1}{2\pi c}\left(\frac{c}{\sigma}\right)^2 \frac{\omega_0^2 - \omega^2}{\omega}$$

This has a double sequence $\{\mp\Omega_n\}$ of solutions. The eigenvectors corresponding to $\Omega_n > 0$ are

$$u_n = A_n \begin{vmatrix} i\dfrac{\sigma}{c}\dfrac{\Omega_n}{\omega_0^2 - \Omega_n^2}\sin\left(\dfrac{\Omega_n l}{c}\right) \\ \sin\dfrac{\Omega_n}{c}(|x| - l) \\ -\dfrac{\sigma}{c}\dfrac{\Omega_n^2}{\omega_0^2 - \Omega_n^2}\sin\left(\dfrac{\Omega_n l}{c}\right) \\ i\Omega_n \sin\dfrac{\Omega_n}{c}(|x| - l) \end{vmatrix}$$

whereas the ones corresponding to $\Omega_n < 0$ are obtained from the u_n's by complex conjugation.

APPENDIX B

Rewrite (2) formally as

$$\xi(t) = e^{Tt}\xi_0 + \int_0^t e^{T(t-s)} W(\xi(s))\, ds \tag{B.1}$$

With the integral a Riemann integral in X (or $\mathcal{H}$). The following partial results sum up in the proof of Theorems 2 and 3.

I. $\forall \xi_0 \in \mathcal{H}$, $\exists T > 0$ such that (A.1) has a unique continuous solution for $t \in J = [0, T]$ with $\xi(0) = \xi_0$.

Proof. Let $X_{T_1\epsilon_1\xi_0}$ be the set of $\xi(t) \in C_0(J, \mathcal{H})$ such that $\xi(0) = \xi_0$ and $\sup_{t\in J}\|\xi(t) - e^{Tt}\xi_0\| \leqslant \epsilon$. This is a complete metric space under the norm of $C_0(J, \mathcal{H})$. Then define S:

$$\xi(t) \rightsquigarrow e^{Tt}\xi_0 + \int_0^t e^{T(t-s)} W(\xi(s))\, ds$$

At least for sufficiently small T, S is a contraction in $X_{T_1\epsilon_1\xi_0}$. This follows from

$$\begin{aligned} \| W(\xi_1) - W(\xi_2)\| &= \alpha\left\||(K_0, \xi_1)|^2(K_0, \xi_1) - |(K_0, \xi_2)|^2(K_0, \xi_2)\right| \\ &\leqslant C(\|\xi_1\|, \|\xi_2\|)\|\xi_1 - \xi_2\| \end{aligned} \tag{B.2}$$

where c is a monotone increasing function of both its arguments.

II. The solution $\xi(t)$ of (B.1) ensured by the contracting mapping principle is certainly a local weak solution of (10) for $t \in J$. Note that $C(\|\xi_0\| + \epsilon_1\|\xi_0\| + \epsilon)$ alone determines how large a T one can choose.

III. Let $\xi(t)$ be a (local) weak solution of (10). Then

$$E(\xi(t)) = \tfrac{1}{2}\|\xi(t)\|^2 + \tfrac{1}{4}\alpha|z|^4 \qquad [z = (K_0, \xi)]$$

is differentiable with respect to t, and $E'(t) = 0$.

Proof. Since $k_0 \in D(T)$, $|z_0^4| = |(K_0, \xi)|^4$ is C^1 in t. Thus one need only prove that $\|\xi\|^2 = \sum|c_n^2(t)|$ is differentiable with respect to t. The series $\sum|c_n^2(t)|$ converges uniformly in J (by Dini's theorem). Looking at the time derivatives

$$\left|\frac{d}{dt}|c_n^2(t)|\right| = \alpha|2\,\mathrm{Re}\{|\varphi(\xi)|^2\bar{\nu}_n\varphi(\xi)\bar{c}_n\}| \leqslant 2\alpha\|\xi\|^3|\nu_n\bar{c}_n|$$

one sees that $\sum(d/dt)|c_n^2(t)|$ converges uniformly in J, since so does $\sum\nu_n\bar{c}_n$. Therefore $\|\xi(t)\|^2$ is differentiable in t and

$$\begin{aligned}\frac{d}{dt}\|\xi\|^2 &= \sum\frac{d}{dt}|c_n^2(t)| = 2\alpha\,\mathrm{Re}\{|\varphi(\xi)|^2\varphi(\xi)(K_1,\xi)\}\\ &= -\frac{d}{dt}\left\{\frac{1}{2}\alpha|\varphi(\xi(t))|^4\right\} = -\frac{d}{dt}\left\{\frac{1}{2}\alpha|z|^4\right\}\end{aligned}$$

IV. (Uniqueness of the local solution.) Let $\xi_1 \cdot \xi_2$ be weak solutions of (10) with $\xi_1(0) = \xi_2(0) = z_0$. By an argument similar to the one of III above, one sees that $\delta(t) = \|\xi_1(t) - \xi_2(t)\|^2$ is differentiable with respect to t, and

$$\delta'(t) = 2\,\mathrm{Re}(\xi_1 - \xi_2, W(\xi_1) - W(\xi_2))$$

whence $|\delta'(t)| \leqslant 2C_\epsilon\delta$ where $C_\epsilon = C(\|\xi_0\| + \epsilon_1\|\xi_0\| + \epsilon)$ was defined in I. Thus $\delta(0) = 0$ implies $\delta(t) = 0$.

V. (Existence of global solutions.)

Proof. This is a standard argument, relying on the a priori boundedness of the local solution stated in III. Let $\bar{T}$ be the sup of the T's such that the argument of I holds in $[0, T]$. Since T is determined by C_ξ alone, and since $\|\xi(t)\|$ is bounded by E, by taking $\xi_0 = \xi(t)$ with t sufficiently near to $\bar{T}$ one could extend the solution beyond $\bar{T}$, whence $\bar{T} = \infty$.

VI. The group property follows as usual from the uniqueness of the solution. From Eqs. (A.1), (A.2)

$$\begin{aligned}\|U_t\xi_1 - U_t\xi_2\| &\leqslant \|\xi_1 - \xi_2\| + \int_0^t \|W(\xi_1(s)) - W(\xi_2(s))\|\,ds\\ &\leqslant \|\xi_1 - \xi_2\| + C(\|\xi_1(t)\|, \|\xi_2(t)\|)\int_0^t \|\xi_1(s) - \xi_2(s)\|\,ds\end{aligned}$$

and, since $\|\xi_1(t)\|^2$ and $\|\xi_2(t)\|^2$ are bounded by E, E_2:

$$\|\xi_1(t) - \xi_2(t)\| \leqslant \|\xi_1(0) - \xi_2(0)\| e^{C(E_1, E_2)t} \tag{B.3}$$

This proves Theorem 3. ■

APPENDIX C

The operator T extends straightforwardly to a skew-adjoint operator in $\mathcal{H}_{-1}$. The same is true for W with the definition (10). The operators in $\mathcal{H}_{-1}$ will be denoted in the same way as the operators in $\mathcal{H}$. The evolution operator $\hat{U}_t x_0 = x(t)$ satisfies an equation formally identical to (B.1), where operators, and the Riemann integral, are to be understood in the $\mathcal{H}_{-1}$ sense. In fact, as $\xi \in \mathcal{H}$ approaches $x \in \mathcal{H}_{-1}$ in the $\mathcal{H}_{-1}$-norm, $U_t\xi$ approaches $\hat{U}_t x$ uniformly on bounded subsets of the line.

Since the same is true for $e^{Tt}z, e^{Tt}x$, and since W is uniformly continuous on bounded subsets of $\mathcal{H}_{-1}$, the above statement follows.

$U_N(t)$ defined by the solutions of (20) satisfies

$$x_N(t) + U_N(t)x_0 = e^{T_N t}x_0 + \int_0^t e^{T_n(t-s)} W_n(x_N(s))\, ds \tag{C.1}$$

where $T_N = P_N T P_N$, $W_n = P_N W P_N$, P_N being the projector (in $\mathcal{H}_{-1}$) onto the phase space of N oscillators. Since P_N is a spectral projection for T, $T_N \to T$ as $N \to \infty$ in the strong resolvent sense, and so $e^{T_N t}x \to e^{Tt}x$, $\forall x \in \mathcal{H}_{-1}$ uniformly for t in bounded intervals. Therefore, for $t \in [0, \theta]$, $\theta > 0$,

$$\begin{aligned}
\| U_N(t)x_0 - \tilde{U}_t x_0\|_{-1} &\leqslant \|e^{T_N t}x_0 - e^{Tt}x_0\|_{-1} \\
&\quad + \int_0^t \|e^{T_N(t-s)} W(x(s)) - e^{T(t-s)} W(x(s))\|_{-1}\, ds \\
&\quad + \int_0^t \| W(x(s)) - W_N(x_N(s))\|_{-2}\, ds \\
&\leqslant \gamma_N^{(1)}(t) + \gamma_N^{(2)}(t) \\
&\quad + \int_0^t \| W(x(s)) - P_N W P_N(x(s))\|_{-1}\, ds \\
&\quad + \int_0^t \| W P_N(x(s)) - W P_N(x_N(s))\|_{-1}\, ds \\
&\leqslant \gamma_N^{(1)}(t) + \gamma_N^{(2)}(t) + \gamma_N^{(3)}(t) + K_r \int_0^t \|x(s) - x_N(s)\|_{-1}\, ds
\end{aligned}$$

$\gamma_N^{(i)}(t)$, $i = 1, 2, 3$, are continuous, equibounded and infinitesimal as $N \to \infty$, for all $t \leqslant \theta$. This yields

$$\|U_N(t)x_0 - \tilde{U}_t x_0\|_{-1} \leqslant \int_0^t e^{K_r(t-s)}\left[\gamma_N^{(1)} + \gamma_N^{(2)} + \gamma_N^{(3)}\right](s)\,ds$$

for all $t < \theta$, and thus $\lim_{N\to\infty} U_N(t)x_0 = \tilde{U}_t x_0$.

APPENDIX D

Let s be the vector space of rapidly decreasing complex sequences $z = \{c_n\}$—i.e., of sequences such that $\forall k\ \|z\|_k^2 = \sum_n \Omega^{2k}|c_n^2| < \infty$—with the Fréchet topology induced by the norms $\|\cdot\|_k$. The measure of $\mu_{N,\sigma}$, as a measure on the dual space s' of s ($s' = \bigcup_{n=1}^{\infty}\mathcal{H}_{-n}$ after an obvious identification) has the characteristic functional[16]

$$M_{N,\sigma}(z) = \left\{c_N \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dq_{-N}\cdots dq_N\, dp_{-N}\cdots dp_N\, w_N(q_{-N},\ldots p_N)\right.$$

$$\left.\times \exp\left[i\left(\sum_{-N}^{+N} \xi_k p_k + \Omega_k^2 \eta_k q_k\right)\right]\right\} \prod_{|j|\geqslant N+1}^{\infty} M_0(\xi_j, \eta_j)$$

M_0 being the c.f. of w, and $w_N = \exp\{-\sigma^{-2}H_N(q_{-N},\ldots p_N)\}$, $z \in s$, $c_n(z) = \xi_n + i\Omega_n\eta_n$.

The following manipulations are justified by the fast-decreasing character of w (the infinite product of M_0's is omitted for simplicity):

$$M_{N,\sigma}(z) = (\cdots)c_N \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dq_{-N}\cdots dp_N\, d\phi\, d\psi$$

$$\times\ \delta\left(\phi - \sum_{-N}^{+} \lambda_k p_k\right)\delta\left(\psi - \sum_{-N}^{+N}\lambda_k\Omega_k q_k\right)\exp\left[i\sum_{-N}^{+N}\left(\xi_K p_K + \Omega_K^2\eta_k q_K^2\right)\right]$$

$$\times \exp\left[-\frac{1}{2\sigma^2}\sum_{-N}^{+N}\left(p_k^2 + \Omega_k^2 q_k^2\right)\right]\exp\left[-\frac{1}{2\omega\sigma^2}\left(\phi^2+\psi^2\right)^2\right]$$

$$= (\cdots)(2\pi)^{-1}c_N \int_{-\infty}^{+\infty} d\gamma \int_{-\infty}^{+\infty} d\theta \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dq_{-N}\cdots dp_N\, d\phi\, d\psi$$

$$\times \exp\left[i\gamma\left(\phi - \sum_{-N}^{+N}\lambda_k p_k\right)\right]\exp\left[i\theta\left(\psi - \sum_{-N}^{+N}\lambda_k q_k \Omega_k\right)\right]$$

$$\times \exp\left[i\sum_{-N}^{+N}\left(\xi_k p_k + \Omega_k^2 \eta_k q_k\right)\right.$$

$$\left. - \frac{1}{2\sigma^2}\sum_{-N}^{+N}\left(p_k^2 + \Omega_k^2 q_k^2\right) - \frac{1}{2\omega\sigma^2}\left(\phi^2+\psi^2\right)^2\right]$$

$$= (\cdots)(2\pi)^{-1}c_N\int_{-\infty}^{+\infty}d\gamma\int_{-\infty}^{+\infty}d\theta$$

$$\times\prod_{k=-N}^{+N}\left\{\int_{-\infty}^{+\infty}dq\exp\left(-\frac{1}{2\sigma^1}\Omega_k^2q^2\right)\exp\left[iq\left(\Omega_k^2\eta_k-\theta\lambda_k\Omega_k\right)\right]\right\}$$

$$\times\left(\int_{-\infty}^{+\infty}dp\exp\left(-\frac{p^2}{2\sigma^2}\right)\exp\left[ip(\xi_k-\gamma\lambda_k)\right]\right)$$

$$\cdot\int_{-\infty}^{+\infty}d\phi\int_{-\infty}^{+\infty}d\psi\exp\left[-\frac{(\phi^2+\phi^2)^2}{2\sigma^2}\right]\exp\left[i(\gamma\phi+\theta\psi)\right]$$

$$= (\cdots)c_N\int_{-\infty}^{+\infty}d\gamma\int_{-\infty}^{+\infty}d\theta\, M(\gamma,\theta)\frac{\sigma^{4N}}{\prod_{j=1}^N\Omega_j^2}$$

$$\times\exp\left[-\frac{\sigma^2}{2}\sum_{-N}^{+N}\left(\theta\lambda_k-\Omega_k^2\eta_k\right)^1\right]\exp\left[-\frac{\sigma^2}{2}\sum_{-N}^{+N}(\xi_k-\gamma\lambda_k)^2\right]$$

$M(\gamma,\theta)$ is the Fourier transform of $\exp[-(x^2+y^2)^2]$. Normalization requires that $M_{N,\sigma}(0)=1$, so that

$$c_N=\left\{\int_{-\infty}^{+\infty}d\gamma\int_{-\infty}^{+\infty}d\theta\, M(\gamma,\theta)\frac{\sigma^2}{\prod_{j=1}^N\Omega_j^2}\exp\left[-\frac{\sigma^2}{2}(\theta^2+\gamma^2)\sum_{-N}^{+N}\lambda_k^2\right]\right\}^{-1}$$

In the limit $N\to\infty$,

$$\lim_{N\to\infty}M_{N,\sigma}(z)=M_\sigma(z)$$

$$=c\int_{-\infty}^{+\infty}d\gamma\int_{-\infty}^{+\infty}d\theta\, M(\gamma,\theta)$$

$$\times\exp\left[-\frac{\sigma^2}{2}\|(\gamma+i\theta)k_0-z\|_0^2\right]$$

$$c=\left\{\int_{-\infty}^{+\infty}d\gamma\int_{-\infty}^{+\infty}d\theta\, M(\gamma,\theta)\exp\left[-\frac{\sigma^2}{2}(\theta^2+\gamma^2)\right]\right\}^{-1}$$

$M_\sigma(z)$ is continuous as a function of $z=\{\xi_k+i\Omega_k\eta_k\}$ in the norm $\|z\|_0$. By Minlos's theorem, it defines a measure μ, on s' concentrated on the set $\sum_j|p_j^2+\Omega_j^2q_j^2|\cdot j^{-2}<+\infty$, that is on $\mathcal{H}_{-1}$. To prove U_t invariance of μ_σ, let $f: \mathcal{H}_{-1} \to \mathbb{C}$ be a bounded uniformly continuous function. Then

$$\begin{aligned}
&\left| \int_{\mathcal{H}_{-1}} f(\tilde{U}_t x)\, d\mu_\sigma(x) - \int_{\mathcal{H}_{-1}} f(x)\, d\mu_\sigma(x) \right| \\
&\quad \leqslant \int_{\mathcal{H}_{-1}} |f(\hat{U}_t x) - f(U_N(t)x)|\, d\mu_\sigma \\
&\quad + \left| \int_{\mathcal{H}_1} f(U_N(t)x)\, d\mu_{k,\sigma} - \int_{\mathcal{H}_{-1}} f(U_N(t)x)\, d\mu_\sigma \right| \\
&\quad + \int_{\mathcal{H}_{-1}} |f(x) - f(U_N(t)x)|\, d\mu_{k,\sigma} + \left| \int_{\mathcal{H}_{-1}} f(x)\, d\mu_{k,\sigma} - \int_{\mathcal{H}_{-1}} f(x)\, d\mu_\sigma \right|
\end{aligned}$$

The second and fourth terms above are $< \epsilon$, $\forall N$, $K > K_\epsilon$ since $\mu_{K,\sigma} \to_{K\to\infty} \mu_\sigma$ entails that $\int g_N\, d\mu_{K,\sigma} \to \int g_N\, d\mu_\sigma$ uniformly on equibounded, equicontinuous[9] families $\{g_N(x)\}$ and $g_N(x) = f(U_N(t)x)$ just make up one such family [thanks to the equicontinuity of $U_N(t)$, that can be inferred from (B.3), and to the uniform continuity of f]. The limits $N \to \infty$ of the first and third term vanish, by the result of Appendix C and dominated convergence.

Therefore the left-hand side of the above inequality is $< \epsilon$, $\forall \epsilon > 0$ and thus it must be zero. This is enough to prove Proposition 3.

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